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ECE 204 *Numerical methods*

An implicit Newton's method

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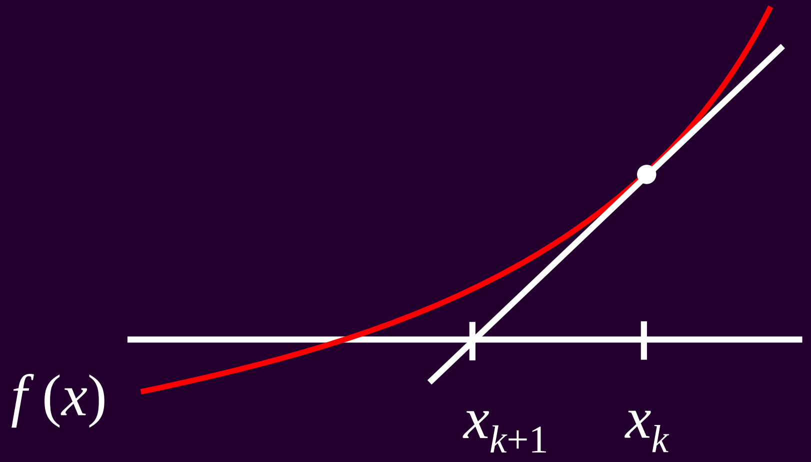
Introduction

- In this topic, we will
  - Recall a weakness of Newton’s method
  - Describe the difference between explicit and implicit
  - Create an implicit version of Newton’s method
  - Give an example where it proves useful
  - Discuss when to use one versus the other

Explicit calculations

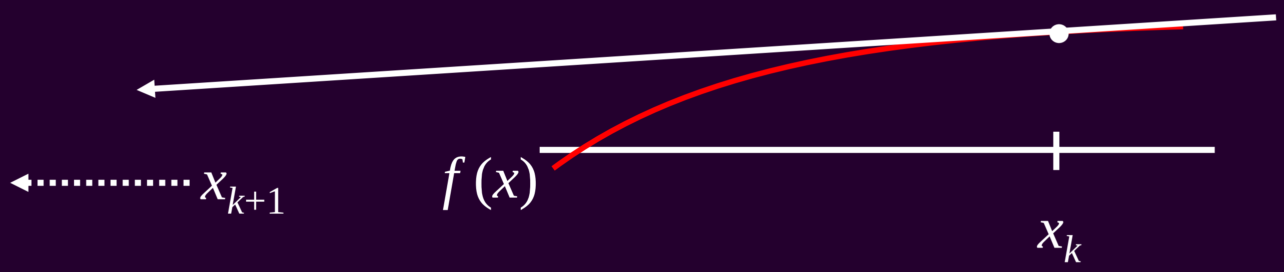
- Newton's method is described as an "explicit" algorithm
  - This is because each subsequent estimate is *explicitly* calculated based on the previous estimate

$$x_{k+1} \leftarrow x_k - \frac{f(x_k)}{f^{(1)}(x_k)}$$



Explicit calculations

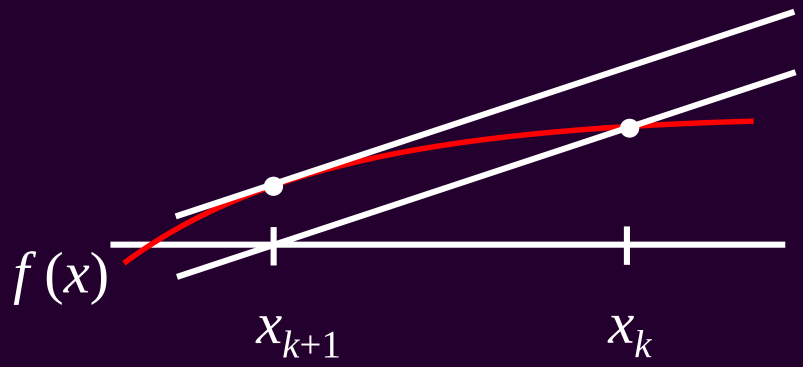
- There is, however, one weakness to Newton's method:
  - If the derivative is close to zero,
    x_{k+1} will be further away from x_k than $|f(x_k)|$
  - If we're never-the-less close to a root,
    we may not converge to that particular root!
  - Given sin(x), if x_0 = 1.6, this is closest to the root at π
    - Newton's method, however, converges to the root at 10π



Implicit questions

- We will instead ask the following question
  - What nearby point x_{k+1} has the a slope $f^{(1)}(x_{k+1})$ such that:
    following that slope from $(x_{k+1}, 0)$
    would go through the point $(x_k, f(x_k))$?

$$f^{(1)}(x_{k+1})(x_k - x_{k+1}) = f(x_k)$$



Implicit Newton’s method

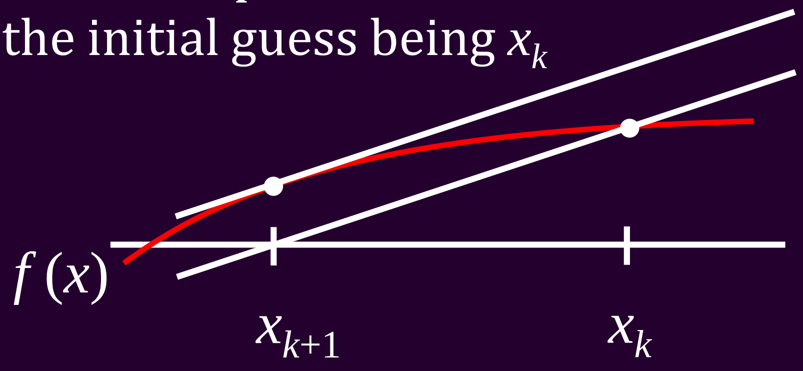
- Unfortunately, while x_k and $f(x_k))$ are known, we cannot rewrite this equation in the form

$$x_{k+1} \leftarrow \text{some expression in } x_k$$

  - The best we can do is

$$x_{k+1} = x_k - \frac{f(x_k)}{f^{(1)}(x_{k+1})}$$

  - In other words, we need to find a solution to $x = x_k - \dfrac{f(x_k)}{f^{(1)}(x)}$
    - We can use fixed-point iteration with the initial guess being x_k



Example

- Suppose we are trying to find the root of sin(x), but x_0 = 1.6
  - In this case, using Newton's method, x_1 = 35.832532736, which is not at all close to 3.141592654
  - Instead, we will use fixed-point iteration, starting with x = 1.6:

$$x = 1.6 - \frac{\sin(1.6)}{f^{(1)}(x)}$$

35.832532736
5.029201577
–1.608498492
28.118647739
2.611811082
2.758365237
2.677750747
2.717665863
2.696648110
2.707395700

Example

- Fixed-point iteration converges slower than Newton's method, yet $x_1 = 2.707\cdots$ is a better approximation of π than $35.832\cdots$
  - For x_2, we can revert to using the explicit Newton's method:

$x_0 = 1.6$

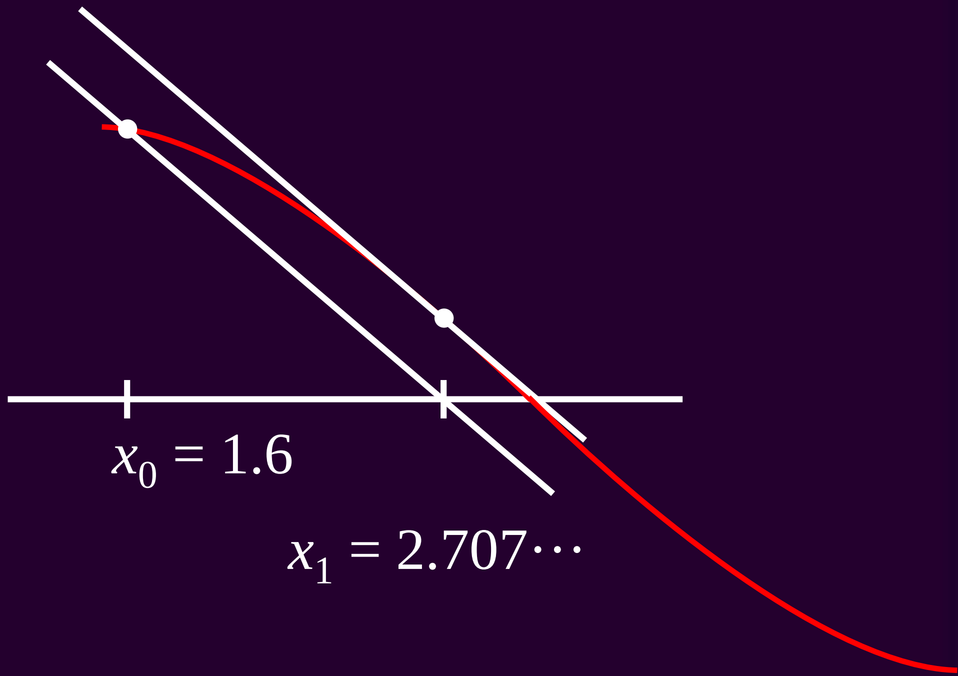
$x_1 = 2.707395699679356$

$x_2 = 3.171106262288358$

$x_3 = 3.141584081296244$

$x_4 = 3.141592653589793$

$x_5 = 3.141592653589793$



Using an implicit Newton's method

- All algorithms we've seen—and most we will see—are explicit
  - Explicit algorithms tend to be faster (nothing to solve)
  - Implicit algorithms can prove valuable when explicit methods encounter difficulties or fail to converge

- Explicit: calculate $x_{k+1} \leftarrow x_k - \dfrac{f(x_k)}{f^{(1)}(x_k)}$
  - Use when $|f^{(1)}(x_k)| \geq 0.5$

- Implicit: solve $x_{k+1} = x_k - \dfrac{f(x_k)}{f^{(1)}(x_{k+1})}$
  - Use when $|f^{(1)}(x_k)| < 0.5$

Summary

- Following this topic, you now
  - Have reviewed Newton's method
  - Understand there can be issues when the derivative is small
  - Know the difference between implicit and explicit algorithms
  - Have seen an implicit version of Newton's method
  - Understand that it is slower than Newton's method,
    but it can also converge to a local root
    when Newton's method does not

References

[1] https://en.wikipedia.org/wiki/Newton%27s_method

[2] https://en.wikipedia.org/wiki/Explicit_and_implicit_methods

Acknowledgments

None so far.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`. Mathematical equations are prepared in MathType by Design Science, Inc.

Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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for more information.

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